Lecture Note 28, (Text book page 1075) 11/05/2018

16.3 The Fundamental Theorem for Line Integrals

Recall the Fundamental Theorem of Calculus can be written as

$$\int_a^b F'(x)dx = F(b) - F(a)$$

where F' is continuous on $[a, b]$.

Theorem 1 Let C be a smooth curve given by the vector function $\mathbf{r}(t)$, $a \leq t \leq b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C. Then

$$\int_C \nabla f \cdot \mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

Definition A **conservative vector field** is a vector field whose line integral over any curve depends only on the endpoints of the curve.

Theorem 1 says that we can evaluate the line integral of a conservative vector field simply by knowing the value of f at the endpoints of C.

Definition A region D is called **open** if D doesn't contain any of its boundary points. In addition, a region D is called **connected** if any two points in D can be joined by a path that lies in D. A curve is called **simple curve** if it does not intersect itself anywhere between its end points. A **simply-connected region** in the plane is a connected region D such that every simple closed curve in D enclose only points are in D.

Theorem 2 Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ be a vector field on an open simply-connected region D. Suppose that P and Q have continuous first-order partial derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \text{throughtout} \quad D$$

Then $\mathbf{F}$ is conservative.

Example I Determine whether or not $\mathbf{F}$ is a conservative vector field. If it is, find a function f such that $\mathbf{F} = \nabla f$.

1. $\mathbf{F}(x, y) = (xy + y^2)\mathbf{i} + (x^2 + 2xy)\mathbf{j}$.
2. $\mathbf{F}(x, y) = ye^x\mathbf{i} + (e^x + e^y)\mathbf{j}$.

Example II (i) Find a function f such that $\mathbf{F} = \nabla f$ and (ii) use part (i) to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the given curve C.

1. $\mathbf{F}(x, y) = (3 + 2xy^2)\mathbf{i} + 2x^2y\mathbf{j}$, C is the arc of the hyperbola $y = 1/x$ from $(1, 1)$ to $(4, 1/4)$.
2. $\mathbf{F}(x, y, z) = (y^2z + 2xz^2)\mathbf{i} + 2xyz\mathbf{j} + (xy^2 + 2x^2z)\mathbf{k}$, $C:\ x = \sqrt{t},\ y = t + 1,\ z = t^2,\ 0 \leq t \leq 1$.

Example 5 Show that the line integral $\int_C \sin y dx + (x \cos y - \sin y) dy$ is independent of the any path C from $(2, 0)$ to $(1, \pi)$ and evaluate the integral.

Homework 4, 7, 9, 13, 15, 17, 19

16.4 Green's Theorem

Green's Theorem gives the relationship between a line integral around a simple closed curve C and a double integral over the plane region D bounded by C.

Definition The **positive orientation** of a simple closed curve C refers to a single *counterclockwise* traversal of C.

Thus if C is given by the vector function $\mathbf{r}(t), a \leq t \leq b$, then the region D is always on the left as the point $\mathbf{r}(t)$ traverses C.

Green's Theorem: Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\int_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

NOTE: The notation $\oint_C P dx + Q dy$ is sometimes used to indicate that the line integral is calculated using the positive orientation of the closed curve C.

Example 1 Evaluate the line integral by two methods: (i) directly and (ii) using Green's Theorem.

1. $\oint_C y dx - x dy$, C is the circle with center the origin and radius 4.
2. $\oint_C (y + e^{\sqrt{x}}) dx + (2x + \cos y^2) dy$, C is the boundary of the region enclosed by the parabolas $y = x^2$ and $x = y^2$.

In previuous examples we found that the double integral was easier to evaluate than the line integral. But some times it's easier to evaluate the line integral, and Green's theorem is used in the reverse direction.

Example 3 What happens if it is known that $P(x, y) = Q(x, y) = 0$ on the curve C but not interior of enclosed region?

An application of Green's theorem is in the computing areas. Since the area of a region D is $\iint_D 1 dA$, we wish to choose P and Q so that $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = 1$. There are several possibilities:

$P(x, y) = 0$	$P(x, y) = -y$	$P(x, y) = -\frac{1}{2}y$
$Q(x, y) = x$	$Q(x, y) = 0$	$Q(x, y) = \frac{1}{2}x$

Example 4 Find the area enclosed by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

Homework 3, 5, 9, 11, 12